

# Spinor representations of $U_q(\hat{\mathfrak{o}}(N))$

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abstract

This is an extension of quantum spinor construction of  $U_q(\hat{\mathfrak{gl}}(n))$ . We define quantum affine Clifford algebras based on the tensor category and the solutions of q-KZ equations, and construct quantum spinor representations of  $U_q(\hat{\mathfrak{o}}(N))$ .

## I. Introduction.

For affine Lie algebras, we have quadratic constructions of representations with Clifford, Weyl or Heisenberg algebras, which are spinor or oscillator representations [FF][KP]. One question is that if we can extend those constructions to the cases for quantum groups, a q-deformation of universal enveloping algebras of Kac-Moody algebras discovered by Drinfeld [D1] and Jimbo [J1]. Lusztig [L] showed the existence of such a q-deformation of the category of highest weight representations of Kac-Moody algebras for generic parameter q. There also appeared many constructions of representations [B], [FJ], [H], [J2], etc.

In [DF2] [Di], we proposed an invariant approach to deform explicitly those classical quadratic constructions with a q-analogue of matrix realization of classical Lie algebras. We managed to use such an approach to define quantum Clifford and Weyl algebras and quantum affine Clifford algebras using general representation theory of quantum groups and solutions of q-kz equations [DO]. We showed that the explicit formulas for quantum Clifford and Weyl algebras match the ones actively studied in physics literature. Using those quantum algebras, we constructed spinor and oscillator representations of quantum groups of classical types and spinor representations of  $U_q(\hat{\mathfrak{gl}}(n))$ . Representation theory allows us to justify that the quadratic elements in quantum Clifford and Weyl algebras and quantum affine Clifford al-

Serre's relations for quantum groups is not necessary. The key idea consists of reformulating familiar classical constructions entirely in terms of the tensor category of highest weight representations and using Lusztig's result on  $q$ -deformation of this category and solutions of  $q$ -kz equations to define the corresponding quantum structures. In the quantum case, the quasitriangular structure of the tensor category introduced by Drinfeld [D3] plays the fundamental role. We would like to stress the central role of the universal Casimir operator and its inverse implied by the quantum structure [Di].

In this paper, we extend the idea in [Di] to the cases of the spinor representations of quantum affine groups  $U_q(\hat{\mathfrak{o}}(N))$ , which is simpler than that in [Di] due to the structure of the corresponding modules. From what we have obtained, it is convincing that we can deform all the constructions given in [FF] as straightforward extensions of our constructions.

[D1] [J1] Quantum group for  $U_q(\mathfrak{g})$  is defined as an associative algebra generated by  $e_i$ ,  $f_i$  and  $t_i$ ,  $i = 0, 1, \dots, n$ , where  $i$  corresponds to the index of the nodes of the Cartan matrix (see Definition 1.1 in Section 1). This algebra has a noncocommutative Hopf algebra structure with comultiplication  $\Delta$ , antipode  $S$  and counit  $\varepsilon$ . [G] The affine Kac-Moody algebra  $\hat{\mathfrak{g}}$  associated to a simple Lie algebra  $\mathfrak{g}$  admits a natural realization as a central extension of the corresponding loop algebra  $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ . Faddeev, Reshetikhin and Takhtajan [FRT2] extended their realization of  $U_q(\mathfrak{g})$  to the quantum loop algebra  $U_q(\mathfrak{g} \otimes [t, t^{-1}])$  via a canonical solution of the Yang-Baxter equation depending on a parameter  $z \in \mathbb{C}$ . The first realization of the quantum affine algebra  $U_q(\hat{\mathfrak{g}})$  and its special degeneration called the Yangian were obtained by Drinfeld [D2]. Later Reshetikhin and Semenov-Tian-Shansky [RS] incorporated the central extension in the previous construction of [FRT1] to obtain another realization of the quantum affine algebra  $U_q(\hat{\mathfrak{g}})$ .

There is an automorphism  $D_z$  of  $U_q(\hat{\mathfrak{g}})$  defined as  $D_z(e_0) = ze_0$ ,  $D_z(f_0) = z^{-1}f_0$ , and  $D_z$  fixes all other generators, where  $e_0$  and  $f_0$  are generators corresponding to the zero node of its Dynkin diagram. We also define the map  $\Delta_z(\varepsilon) = (D_z \otimes$

$id)\Delta(a)$  and  $\Delta'_z(a) = (D_z \otimes id)\Delta'(a)$ , where  $a \in U_q(\hat{\mathfrak{g}})$  and  $\Delta'$  denotes the opposite comultiplication. Let  $d$  be an operator such that  $d$  commutes with all other elements but has the relations  $[d, e_0] = e_0, [d, f_0] = -f_0$ . It is clear that the action of  $D_z$  is equivalent to the conjugation by  $z^d$ . For the algebra generated by  $U_q(\hat{\mathfrak{g}})$  and the operator  $d$  which we denote by  $U_q(\tilde{\mathfrak{g}})$ , from the theory of Drinfeld[D2], we know that it has a universal R-matrix  $\bar{\mathfrak{R}}, \bar{\mathfrak{R}}$  in  $U_q^+(\tilde{\mathfrak{g}}) \hat{\otimes} U_q^-(\tilde{\mathfrak{g}})$  such that  $\bar{\mathfrak{R}}$  satisfies the properties:

$$\begin{aligned} \bar{\mathfrak{R}}\Delta(a) &= \Delta^{\text{op}}(a)\bar{\mathfrak{R}}, \\ (\Delta \otimes \text{id})(\bar{\mathfrak{R}}) &= \bar{\mathfrak{R}}_{13}\bar{\mathfrak{R}}_{23}, \\ (\text{id} \otimes \Delta)(\bar{\mathfrak{R}}) &= \bar{\mathfrak{R}}_{13}\bar{\mathfrak{R}}_{12}, \end{aligned} \tag{I.1}$$

where  $a \in U_q(\mathfrak{g})$ ,  $\Delta^{\text{op}}$  denotes the opposite comultiplication,  $\bar{\mathfrak{R}}_{12} = \sum_i a_i \otimes b_i \otimes 1 = \bar{\mathfrak{R}} \otimes 1, \bar{\mathfrak{R}}_{13} = \sum_i a_i \otimes 1 \otimes b_i, \bar{\mathfrak{R}}_{23} = \sum_i 1 \otimes a_i \otimes b_i = 1 \otimes \bar{\mathfrak{R}}$ . Here  $U_q^+(\tilde{\mathfrak{g}})$  is the subalgebra generated by  $e_i, t_i$  and  $d$  and  $U_q^-(\tilde{\mathfrak{g}})$  is the subalgebra generated by  $f_i, t_i$  and  $d$ .

Let  $\bar{\mathfrak{R}}(z) = (D_z \otimes \text{id})\bar{\mathfrak{R}}$ . Let  $C$  be the central element corresponding to the central extension of the quantum affine algebra. Let  $\mathfrak{R}(z) = q^{-d \otimes C - C \otimes d} \bar{\mathfrak{R}}(z)$ , then we have  $\mathfrak{R}(z) \in U_q(\hat{\mathfrak{b}}^+) \hat{\otimes} U_q(\hat{\mathfrak{b}}^-) \otimes \mathbb{C}[[z]]$ , such that

$$\begin{aligned} \mathfrak{R}(z)\Delta_z(a) &= (D_{q^{C_2}}^{-1} \otimes D_{q^{C_1}}^{-1})\Delta'_z(a)\mathfrak{R}(z), \\ (\Delta \otimes I)\mathfrak{R}(z) &= \mathfrak{R}_{13}(zq^{C_2})\mathfrak{R}_{23}(z), \\ (I \otimes \Delta)\mathfrak{R}(z) &= \mathfrak{R}_{13}(zq^{-C_2})\mathfrak{R}_{12}(z), \end{aligned} \tag{I.2}$$

$$\mathfrak{R}_{12}(z)\mathfrak{R}_{13}(zq^{C_2}/w)\mathfrak{R}_{23}(w) = \mathfrak{R}_{23}(w)\mathfrak{R}_{13}(zq^{-C_2}/w)\mathfrak{R}_{12}(z).$$

Here  $C_1 = C \otimes 1, C_2 = 1 \otimes C, U_q(\hat{\mathfrak{b}}^+)$  is the subalgebra generated by  $e_i, t_i$  and  $U_q(\hat{\mathfrak{b}}^-)$  is the subalgebra generated by  $f_i, t_i$ .

Let  $\mathfrak{C} = ((D_{q^{C_2}}^{-1} \otimes D_{q^{C_1}}^{-1})\mathfrak{R}_{21})\mathfrak{R}$ . Then

$$\sigma \Delta(a) = ((D_{q^{C_2}}^{-1} \otimes D_{q^{C_1}}^{-1}) \Delta(a)) \sigma \tag{I.3}$$

We note that  $(q^{C_2} \otimes q^{C_1})$  is invariant under the permutation. This shows that the action of  $\mathfrak{C} = ((D_{q^{C_2}}^{-1} \otimes D_{q^{C_1}}^{-1})\mathfrak{R}_{21})\mathfrak{R}$  on a tensor product of two modules is an intertwiner which, however, shifts the actions of  $e_0$  and  $f_0$  by the constants  $q^{\mp 2C_2} \otimes q^{\mp 2C_1}$  respectively. We should also notice that  $D_z \otimes D_z$  acts invariantly on  $\mathfrak{R}$ . Let  $\mathfrak{R}_{21}(z) = (D_z \otimes 1)\mathfrak{R}_{21}$ . Note that  $\mathfrak{R}_{21}(z)$  is not equal to  $P(\mathfrak{R}(z))$ , where  $P$  is the permutation operator.

Let  $V$  be a finite dimensional representation of  $U_q(\hat{\mathfrak{g}})$ . Let  $\bar{L}^+(z) = (\text{id} \otimes \pi_V)(\mathfrak{R}_{21}(z))$ ,  $\bar{L}^{-1}(z) = (\text{id} \otimes \pi_V)\mathfrak{R}(z^{-1})$ ,  $\mathfrak{L}^+(z) = (\pi_V \otimes \text{id})(\mathfrak{R}^{-1}(z))$ ,  $\mathfrak{L}^{-1}(z) = (\pi_V \otimes \text{id})\mathfrak{R}_{21}^{-1}(z^{-1})$ . We know that  $\bar{L}^{-1}(z)P\mathfrak{L}^{-1}(z^{-1}) = 1$ , where  $P$  is the permutation operator.

$U_q(\hat{\mathfrak{g}})$  as an algebra is generated by operator entries of  $\bar{L}^+(z)$  and  $\bar{L}^{-1}(z)$ , and it is also generated by operator entries of  $\mathfrak{L}^+(z)$  and  $\mathfrak{L}^{-1}(z)$ .  $\bar{L}^\pm(z)$  are used in [FR] to obtained q-KZ equation. However,  $\bar{L}^\pm(z)$  and  $\mathfrak{L}^\pm(z)$  are basically the same.

Let

$$\begin{aligned}\bar{L}(z) &= (\text{id} \otimes \pi_V)((1 \otimes D_{q^C}^{-1})\mathfrak{R}_{21}(z))\mathfrak{R}(z^{-1}), \\ (D_z \otimes 1)\bar{L}(z) &= \bar{L} = (\text{id} \otimes \pi_V)((1 \otimes D_{q^C}^{-1})\mathfrak{R}_{21})\mathfrak{R}, \\ \mathfrak{L}(z) &= (\pi_V \otimes \text{id})((D_{q^C} \otimes 1)\mathfrak{R}^{-1}(z))\mathfrak{R}_{21}^{-1}(z^{-1}), \\ (1 \otimes D_z)\mathfrak{L}(z) &= \mathfrak{L} = (\pi_V \otimes \text{id})((1 \otimes D_{q^C})\mathfrak{R}^{-1})\mathfrak{R}_{21}^{-1}.\end{aligned}$$

Then

$$\begin{aligned}(\text{I.4}) \quad \bar{R}\left(\frac{z}{w}\right)\mathfrak{L}_1^\pm(z)^{-1}\mathfrak{L}_2^\pm(w)^{-1} &= \mathfrak{L}_2^\pm(w)^{-1}\mathfrak{L}_1^\pm(z)^{-1}\bar{R}\left(\frac{z}{w}\right), \\ \bar{R}\left(\frac{zq^{-C}}{w}\right)\mathfrak{L}_1^+(z)^{-1}\mathfrak{L}_2^-(w)^{-1} &= \mathfrak{L}_2^-(w)^{-1}\mathfrak{L}_1^+(z)^{-1}\bar{R}\left(\frac{zq^C}{w}\right), \\ \mathfrak{L}_1(z)\bar{R}(zq^{-2C}/w)\mathfrak{L}_2(w)\bar{R}(z/w)^{-1} &= \bar{R}(z/w)\mathfrak{L}_2(w)\bar{R}(zq^{2C}/w)^{-1}\mathfrak{L}_1(z), \\ \mathfrak{L}(z)(\text{id} \otimes \pi_V)\Delta(a) &= (D_{q^{2C}} \otimes 1\Delta(a))\mathfrak{L}(z),\end{aligned}$$

Here  $\bar{R}(z/w)$  is the image of  $\mathfrak{R}(z/w)$  on  $V \otimes V$ .

We name  $\bar{L}(z)$  and  $\mathfrak{L}(z)$  universal Casimir operator and inverse universal Casimir operator of the quantum algebra respectively.

The construction of a representation  $U_q(\hat{\mathfrak{g}})$  is equivalent to finding a specific realization of the operator  $\bar{L}(z)$  or  $\mathfrak{L}(z)$ , which plays the same role as  $L$  in the case of  $U_q(\mathfrak{g})$  in [DF2]. Naturally we would like to find a way to build this  $\bar{L}(z)$  or  $\mathfrak{L}(z)$  out of the intertwiners, just as in the case of spinor and oscillator representations of the quantum groups of types  $A$ ,  $B$ ,  $C$  and  $D$  in [DF2]. For this paper, we will only use  $\mathfrak{L}(z)$ .

Let  $V_{\lambda,k}$  and  $V_{\lambda_1,k}$  be two highest weight representations of  $U_q(\hat{\mathfrak{g}})$  with highest weight  $\lambda$  and  $\lambda_1$  and the center  $C$  acting as a multiplication by a number  $k$ . Let  $\Phi$  be an intertwiner:  $\Phi : V_{\lambda_1,k} \longrightarrow V \otimes V_{\lambda,k}$ ,  $\Phi(x) = E_1 \otimes \Phi_1(x) + \dots + E_n \otimes \Phi_n(x)$ , where  $x \in V_{\lambda_1,k}$  and  $\{E_i\}$  is the basis for  $V$ . Let  $\Phi^*$  be an intertwiner:  $\Phi^* : V_{\lambda,k} \longrightarrow {}^*V \otimes V_{\lambda_1,k}$ ,  $\Phi^*(x) = E_1^* \otimes \Phi_1^*(x) + \dots + E_n^* \otimes \Phi_n^*(x)$ , where  ${}^*V$  is the right dual representation of  $V$  of  $\hat{U}_q(\mathfrak{g})$ ,  $x \in V_{\lambda,k}$  and  $E_i^*$  is the basis for  ${}^*V$ . By the right dual representation of  $V$  of  $\hat{U}_q(\mathfrak{g})$ , we mean the action of  $U_q(\hat{\mathfrak{g}})$  on the dual space given by  $\langle av', v \rangle = \langle v', S^{-1}(a)v \rangle$ , for  $a \in \hat{U}_q(\mathfrak{g})$ ,  $v \in V$  and  $v' \in {}^*V$ .  $(1 \otimes \Phi)\Phi^* = \sum \Phi_i \Phi_j^* \otimes E_j^* \otimes E_i$  gives a map  $\Phi : V_{\lambda,k} \longrightarrow {}^*V \otimes V \otimes V_{\lambda,k}$ .

Let us identify  ${}^*V \otimes V$  with  $\text{End}(V)$  by the map which maps the first two components of  $V \otimes {}^*V \otimes V$  to  $\mathbb{C}$  and fix the last component. Let  $\tilde{\mathfrak{L}} \in \text{End}(V) \otimes \text{End}(V_{\lambda,k})$  be:  $\tilde{\mathfrak{L}} = (\tilde{\mathfrak{L}}_{ij}) = ((D_{q^{2k}}\Phi_j)\Phi_i^*)$ , where  $(D_{q^{2k}}\Phi_i)$  means shifting the evaluation representation by constant  $q^{2k}$  and we assume  $\tilde{\mathfrak{L}}$  is well defined. Then

$$(I.5) \quad \tilde{\mathfrak{L}}\Delta(a) = ((D_{q^{2k}} \otimes 1)\Delta(a))\tilde{\mathfrak{L}}.$$

The key idea is to identify  $\tilde{L}$  with  $L$  or  $\tilde{\mathfrak{L}}$  with  $\mathfrak{L}$  to obtain representations out of intertwiners, which is how we obtained the spinor representations for  $U_q(\hat{\mathfrak{gl}}(n))$  [Di].

This paper is arranged as follows. We prepare the basic results about two point correlation functions based on the solutions of q-KZ equations[DvO] in Section 1. Then in Section 2, we will present the commutation relations of the intertwiners, identify them as quantum affine Clifford algebras, and finally derive spinor representations. We will also discuss its connection with other problems.

In this section, we will give the results on the two point functions of type II intertwiners, which basically come from the results of Davies and Okado [DvO] about the explicit solutions of q-KZ equations of the quantum groups of  $B_n^{(1)}$  and  $D_n^{(1)}$ . We will follow the notations in [DvO].

Let  $\mathfrak{h}^*$ ,  $\Lambda_i, h_i = \alpha_i^\vee, \alpha_i, \delta = \sum_{i=0}^n a_i \alpha_i$  and  $d$  be the standard notations in [K]. The dual Coxeter number is defined as  $h^\vee = \sum_{i=0}^n a_i^\vee$ , where  $a_i^\vee = a_i$  except for  $B_n^{(1)}$ ,  $a_n^\vee = a_n/2$ , so that we have  $h^\vee = 2n - 1$  for  $B_n^{(1)}$  and  $2n - 2$  for  $D_n^{(1)}$ . The invariant bilinear form  $(|)$  in [K] is normalized so that  $(\theta|\theta) = 2$ , where  $\theta = \delta - \alpha_0$ . We put  $\rho = \sum_{i=0}^n \Lambda_i$ . For  $\lambda \in \mathfrak{h}^*$   $\bar{\lambda}$  denotes the orthogonal projection of  $\lambda$  on  $\bar{\mathfrak{h}}^*$ , where  $\bar{\mathfrak{h}}^*$  is the linear span of the classical roots  $\alpha_1, \dots, \alpha_n$ . We introduce an orthonormal basis  $\{\epsilon_1, \dots, \epsilon_n\}$  of  $\bar{\mathfrak{h}}^*$ , by which  $\alpha_i, \bar{\Lambda}_i, \bar{\rho}$  are represented below.

$$\begin{aligned}
(1.1) \quad & \alpha_i = \epsilon_i - \epsilon_{i+1} & (1 \leq i \leq n-1), \\
& = \epsilon_n & (i = n \text{ for } B_n^{(1)}), \\
& = \epsilon_{n-1} + \epsilon_n & (i = n \text{ for } D_n^{(1)}), \\
& \bar{\Lambda}_i = \epsilon_1 + \dots + \epsilon_i & (1 \leq i \leq n-1 \text{ for } B_n^{(1)}, 1 \leq i \leq n-2 \text{ for } D_n^{(1)}), \\
& = \frac{\epsilon_1 + \dots + \epsilon_{n-1} - \epsilon_n}{2} & (i = n-1 \text{ for } D_n^{(1)}), \\
& = \frac{\epsilon_1 + \dots + \epsilon_{n-1} + \epsilon_n}{2} & (i = n), \\
& 2\bar{\rho} = (2n-1)\epsilon_1 + (2n-3)\epsilon_2 + \dots + 3\epsilon_{n-1} + \epsilon_n & (\text{ for } B_n^{(1)}), \\
& = (2n-2)\epsilon_1 + (2n-4)\epsilon_2 + \dots + 2\epsilon_{n-1} & (\text{ for } D_n^{(1)}).
\end{aligned}$$

**Definition 1.1.** *The quantum affine algebra  $U_q(\tilde{\mathfrak{g}})$  is defined as the  $\mathbf{Q}(q)$ -algebra with 1 generated by  $e_i, f_i, t_i (i = 0, \dots, n), q^d$  satisfying*

$$\begin{aligned}
(1.2) \quad & [t, t'] = 0 \quad \text{for } t, t' \in \{t_0, \dots, t_n, q^d\}, \\
& t_i e_j t_i^{-1} = q^{(\alpha_i|\alpha_j)} e_j, \quad t_i f_j t_i^{-1} = q^{-(\alpha_i|\alpha_j)} f_j, \\
& q^d e_j q^{-d} = q^{\delta_{j0}} e_j, \quad q^d f_j q^{-d} = q^{-\delta_{j0}} f_j, \\
& [e_i, f_j] = \delta_{ij} (t_i - t_i^{-1}) / (q_i - q_i^{-1}), \\
& \sum_{k=0}^b (-)^k e_i^{(k)} e_j e_i^{(b-k)} = \sum_{k=0}^b (-)^k f_i^{(k)} f_j f_i^{(b-k)} = 0 \quad (i \neq j),
\end{aligned}$$

where  $b = 1, \dots, b = \infty$ . Here we have set  $q_i = q^{(\alpha_i|\alpha_i)/2}$  [23],  $q_i^{-m} = q^{-m(\alpha_i|\alpha_i)/2}$  [23].

$q_i^{-1}$ ),  $[k]_i! = \prod_{m=1}^k [m]_i$ ,  $e_i^{(k)} = e_i^k / [k]_i!$ ,  $f_i^{(k)} = f_i^k / [k]_i!$ . We denote by  $U_q(\hat{\mathfrak{g}})$  the subalgebra of  $U_q(\tilde{\mathfrak{g}})$  generated by the elements  $e_i, f_i, t_i$  ( $i = 0, \dots, n$ ). Let  $x_i$  be any of  $e_i, f_i, t_i$ . The coproduct  $\Delta$  and the antipode  $a$  are defined as follows.

$$\begin{aligned}\Delta(e_i) &= e_i \otimes 1 + t_i \otimes e_i, & \Delta(f_i) &= f_i \otimes t_i^{-1} + 1 \otimes f_i, \\ \Delta(t_i) &= t_i \otimes t_i, & \Delta(q^d) &= q^d \otimes q^d, \\ a(e_i) &= -t_i^{-1}e_i, & a(f_i) &= -f_i t_i, & a(t_i) &= t_i^{-1}, & a(q^d) &= q^{-d}.\end{aligned}$$

For a representation  $(\pi, V)$  of  $U_q(\hat{\mathfrak{g}})$ , we put  $V_z = V \otimes \mathbf{Q}(q)[z, z^{-1}]$ , and lift  $\pi$  to a representation  $(\pi_z, V_z)$  of  $U_q(\tilde{\mathfrak{g}})$  as follows:  $\pi_z(e_i)(v \otimes z^n) = \pi(e_i)v \otimes z^{n+\delta_{i0}}$ ,  $\pi_z(f_i)(v \otimes z^n) = \pi(f_i)v \otimes z^{n-\delta_{i0}}$ ,  $\pi_z(t_i)(v \otimes z^n) = \pi(t_i)v \otimes z^n$ ,  $\pi_z(q^d)(v \otimes z^n) = v \otimes (qz)^n$ . Define an index set  $J$  by  $J = \{0, \pm 1, \dots, \pm n\}$ , for  $\mathfrak{g} = B_n^{(1)}$ ;  $J = \{\pm 1, \dots, \pm n\}$ , for  $\mathfrak{g} = D_n^{(1)}$ , and set  $N = |J|$ . We introduce a linear order  $\prec$  in  $J$  by  $1 \prec 2 \prec \dots \prec n (\prec 0) \prec -n \prec \dots \prec -2 \prec -1$ . We shall also use the usual order  $<$  in  $J$ .

We shall define the vector representation  $(\pi^{(1)}, V^{(1)})$  of  $U_q(\hat{\mathfrak{g}})$ . It is the fundamental representation associated with the first node in the Dynkin diagram. The base vectors of  $V^{(1)}$  are given by  $\{v_j \mid j \in J\}$ . The weight of  $v_j$  is given by  $\epsilon_j (j > 0)$ ,  $-\epsilon_{-j} (j < 0)$ ,  $0 (j = 0)$ . We take  $v_1$  as a reference vector. Set  $s = (-)^n$  for  $\mathfrak{g} = B_n^{(1)}$ , and  $s = (-)^{n-1}$  for  $\mathfrak{g} = D_n^{(1)}$ .

Denoting the matrix units by  $E_{-i} = E_{-\alpha_i}$ ,  $E_{\alpha_i} = E_i$ , the actions of the generators

read as follows  $(\pi^{(1)}(f_i) = \pi^{(1)}(e_i)^t)$ :

$$\begin{aligned}
\pi^{(1)}(e_0) &= s(E_{-1,2} - E_{-2,1}), \\
\pi^{(1)}(t_0) &= \sum_{j \in J} q^{-\delta_{j1} - \delta_{j2} + \delta_{j,-1} + \delta_{j,-2}} E_{jj}, \\
\pi^{(1)}(e_i) &= E_{i,i+1} - E_{-i-1,-i} \quad (1 \leq i \leq n-1), \\
\pi^{(1)}(t_i) &= \sum_{j \in J} q^{\delta_{ji} - \delta_{j,i+1} + \delta_{j,-i-1} - \delta_{j,-i}} E_{jj} \quad (1 \leq i \leq n-1), \\
\pi^{(1)}(e_n) &= \sqrt{[2]_n} (E_{n0} - E_{0,-n}) && \text{for } \mathfrak{g} = B_n^{(1)}, \\
&= E_{n-1,-n} - E_{n,-n+1} && \text{for } \mathfrak{g} = D_n^{(1)}, \\
\pi^{(1)}(t_n) &= \sum_{j \in J} q^{\delta_{j,n} - \delta_{j,-n}} E_{jj} && \text{for } \mathfrak{g} = B_n^{(1)}, \\
&= \sum_{j \in J} q^{\delta_{j,n-1} + \delta_{jn} - \delta_{j,-n} - \delta_{j,-n+1}} E_{jj} && \text{for } \mathfrak{g} = D_n^{(1)}.
\end{aligned}$$

Let  $\{v_j^* \mid j \in J\}$  be the canonical dual basis of  $\{v_j \mid j \in J\}$ . Then we have the following isomorphism of  $U_q(\tilde{\mathfrak{g}})$ -modules.

$$C_{\pm}^{(1)} : V_{z\xi^{\mp 1}}^{(1)} \xrightarrow{\sim} (V_z^{(1)})^{*a^{\pm 1}}; \quad v_j \mapsto q^{\pm \bar{j}} v_{-j}^*.$$

Here  $\bar{j}$  is defined by  $\bar{j} = j$ , if  $j > 0$ ; or  $n$ , if  $j = 0$ ; or  $j + N$ , if  $j < 0$ .

We also define a matrix  $\bar{R}^{(1,1)}(z)$  on  $V^{(1)} \otimes V^{(1)}$  as:

$$\begin{aligned}
\bar{R}^{(1,1)}(z) &= \sum_{i \neq 0} E_{ii} \otimes E_{ii} + \frac{q(1-z)}{1-q^2z} \sum_{i \neq \pm j} E_{ii} \otimes E_{jj} \\
(1.3) \quad &+ \frac{1-q^2}{1-q^2z} \left( \sum_{i \prec j, i \neq -j} + z \sum_{i \succ j, i \neq -j} \right) E_{ij} \otimes E_{ji} \\
&+ \frac{1}{(1-q^2z)(1-\xi z)} \sum_{i,j} a_{ij}(z) E_{ij} \otimes E_{-i,-j}.
\end{aligned}$$

Here

$$\begin{aligned}
a_{ij}(z) &= (q^2 - \xi z)(1-z) + \delta_{i0}(1-q)(q+z)(1-\xi z), (i = j); \\
(1.4) \quad &= (1-q^2)(q^{\bar{j}-\bar{i}}(z-1) + \delta_{i,-j}(1-\xi z)), (i \prec j); \\
&= (1-q^2)z(\xi q^{\bar{j}-\bar{i}}(z-1) + \delta_{i,-j}(1-\xi z)), (i \succ j).
\end{aligned}$$

Let  $V(\Lambda_i)$  be the irreducible highest weight module with highest weight  $\Lambda_i$ . Let

$|\Lambda_i\rangle$  be a highest weight vector of  $V(\Lambda_i)$ . We only consider level 1 modules, i.e.



$i = 0, 1, n$  (and  $n - 1$  for  $\mathfrak{g} = D_n^{(1)}$ ). Let  $\lambda, \mu$  stand for level 1 weights. We define intertwiners as the the following:

$$\begin{aligned}\tilde{\Phi}_\lambda^{\mu V^{(1)}}(z) : \quad V(\lambda) &\longrightarrow V(\mu) \otimes V_z^{(1)}, \\ \tilde{\Phi}_\lambda^{V^{(1)}\mu}(z) : \quad V(\lambda) &\longrightarrow V_z^{(1)} \otimes V(\mu).\end{aligned}$$

We call them type I and type II respectively depending on the location of  $V^{(1)}$ .

For the vector representation  $V^{(1)}$  we fix the normalisations as follows:

$$(1.5) \quad \begin{aligned}\tilde{\Phi}_{\Lambda_1}^{V^{(1)}\Lambda_0}(z)|\Lambda_1\rangle &= v_1 \otimes |\Lambda_0\rangle + \cdots & \text{for } \mathfrak{g} = B_n^{(1)}, D_n^{(1)}, \\ \tilde{\Phi}_{\Lambda_n}^{V^{(1)}\Lambda_n}(z)|\Lambda_n\rangle &= \alpha^{-1}v_0 \otimes |\Lambda_n\rangle + \cdots & \text{for } \mathfrak{g} = B_n^{(1)},\end{aligned}$$

where

$$\alpha = \sqrt{[2]_n}.$$

$\tilde{\Phi}_{\Lambda_0}^{V^{(1)}\Lambda_1}$  ( $\mathfrak{g} = B_n^{(1)}, D_n^{(1)}$ ) and  $\tilde{\Phi}_{\Lambda_{n-1}}^{V^{(1)}\Lambda_n}$ ,  $\tilde{\Phi}_{\Lambda_n}^{V^{(1)}\Lambda_{n-1}}$  ( $\mathfrak{g} = D_n^{(1)}$ ) are normalised using Dynkin diagram automorphisms.

As explained in [FF], it is necessary to shift the grading by  $1/2$ , in order to write down simple commutation relations between those operators. We will assume that  $\tilde{\Phi}_\mu^{(V^{(1)})_1\lambda}(z)$ , for  $\mu = \Lambda_{0,1}$ , is shifted by  $z^{\mp 1/2}$  respectively, we normalize them such that

$$(1.6) \quad \begin{aligned}\tilde{\Phi}_{\Lambda_1}^{V^{(1)}\Lambda_0}(z)|\Lambda_1\rangle &= z^{-1/2}v_1 \otimes |\Lambda_0\rangle + \cdots, \\ \tilde{\Phi}_{\Lambda_0}^{V^{(1)}\Lambda_1}(z)|\Lambda_0\rangle &= z^{1/2}v_{-1} \otimes |\Lambda_1\rangle + \cdots.\end{aligned}$$

With the nomalization, from now on, we will use  $\Phi_\lambda^{V^{(1)}\mu}$ , instead of  $\tilde{\Phi}_\lambda^{V^{(1)}\mu}$  to denote those operators.

Let  $\langle \nu | \Phi_\mu^{\nu W_2}(z_2) \Phi_\lambda^{\mu V_1}(z_1) | \lambda \rangle$  be the expectation value of the following composition of the intertwiners

$$(1.7) \quad V(\lambda) \xrightarrow{\Phi_\lambda^{\mu V(z_1)}} V(\mu) \otimes V_{z_1} \xrightarrow{\Phi_\mu^{\nu W(z_2)} \otimes 1} V(\nu) \otimes W_{z_2} \otimes V_{z_1} \xrightarrow{1 \otimes P} V(\nu) \otimes V_{z_1} \otimes W_{z_2}.$$

Recalling the isomorphism  $C_\pm^{(1)}$ , we define

$$(1.8) \quad \Phi_\lambda^{\mu(V^{(1)})^*a\pm 1}(z) = (C_\pm^{(1)} \otimes \text{id}) \cdot \Phi_\lambda^{\mu V^{(1)}}(z\xi^{\mp 1}).$$

**Theorem 1.1.** [DvO] [O] *For any possible combination of weights  $(\lambda, \mu)$  from  $\Lambda_0$ ,  $\Lambda_1$ ,  $\Lambda_{n-1}, \Lambda_n$*

(1.9)

$$\Phi_\mu^{(V^{(1)})_2\lambda}(z_1)\Phi_\lambda^{(V^{(1)})_1\mu}(z_2) = P\rho(z_1/z_2)\bar{R}^{(1,1)}(z_1/z_2)\Phi_\mu^{(V^{(1)})_2\lambda}(z_2)\Phi_\lambda^{(V^{(1)})_1\mu}(z_1),$$

where

$$(1.10) \quad \rho(z) = z \frac{(z^{-1}; \xi^2)_\infty (q^{-2}\xi z; \xi^2)_\infty (\xi z; \xi^2)_\infty (q^{-2}\xi^2 z; \xi^2)_\infty}{(z; \xi^2)_\infty (q^{-2}\xi^2 z; \xi^2)_\infty (\xi z^{-1}; \xi^2)_\infty (q^{-2}\xi z^{-1}; \xi^2)_\infty},$$

and  $P$  is the permutation operator.

The equality of the commutation relations in the theorem is valid on the level of the correlation functions.

## 2. Spinor representations for $B_n^{(1)}$ and $D_n^{(1)}$ .

In [Di], we presented a construction of spinor representations of  $U_q(\hat{\mathfrak{gl}}(n))$ . We will construct spinor representations in a completely parallel way for  $U_q(\hat{\mathfrak{o}}(n))$ , which is a quantization of corresponding classical constructions in [FF]. We will identify the type II intertwiners as generators of quantum affine Clifford algebras and construct the inverse universal Casimir operators.

In this section, we will assume that  $|q| < 1$ .

Let  $\Lambda$  be  $V(\Lambda_n)$  for the case of  $B_n^{(1)}$  and  $\Lambda$  be  $V(\Lambda_{n-1}) \oplus V(\Lambda_n)$  or  $V(\Lambda_0) \oplus V(\Lambda_1)$  for the case of  $D_n^{(1)}$ .

Let  $\Phi(z)$  be the intertwiner  $\Phi_{\Lambda_n}^{V^{(1)}\Lambda_n}$  from  $\Lambda$  to  $V_z^{(1)} \otimes \Lambda$  for the case of  $B_n^{(1)}$  and  $\Phi(z)$  be  $\Phi_{\Lambda_{n-1}}^{V^{(1)}\Lambda_n} \oplus \Phi_{\Lambda_n}^{V^{(1)}\Lambda_{n-1}}(z)$  or  $\Phi_{\Lambda_0}^{V^{(1)}\Lambda_1}(z) \oplus \Phi_{\Lambda_1}^{V^{(1)}\Lambda_0}$  as an intertwiner from  $\Lambda$  to  $V_z^{(1)} \otimes \Lambda$ .

**Theorem 2.1.**  $\Phi(z)$  satisfies the following commutation relations,

$$(A) \quad F(z_1/z_2)\Phi^2(z_1)\Phi^1(z_2) = G(z_1/z_2)P\bar{R}^{(1,1)}(z_1/z_2)\Phi^2(z_2)\Phi^1(z_1) + F\delta(z_1\xi/z_2),$$

when  $\Lambda$  is  $\Lambda_n$  or  $\Lambda_{n-1} \oplus \Lambda_n$ ;

(B)

$$F(z_1/z_2)\Phi^1(z_1)\Phi^2(z_2) = G(z_1/z_2)P\bar{R}^{(1,1)}(z_1/z_2)\Phi^2(z_2)\Phi^1(z_1) + (z_1/z_2)^{1/2}F\delta(z_1\xi/z_2)$$

when  $\Lambda$  is  $\Lambda_0 \oplus \Lambda_1$ .

$$(2.1) \quad F(z) = \frac{(\xi^1 z^{-1}; \xi^2)_\infty (q^{-2} z^{-1}; \xi^2)_\infty}{(z^{-1} \xi^2; \xi^2)_\infty (q^{-2} \xi z^{-1}; \xi^2)_\infty},$$

$$G(z) = \rho(z)F(z),$$

$P$  is the permutation operator, the matrix  $\bar{R}^{(1,1)}(z_1/z_2)$  is defined for  $B_n^{(1)}$  and  $D_n^{(1)}$  respectively as in Section 1, and  $F$  is a nonzero vector in the one dimensional invariant subspace under the action of the subalgebra  $U_q(\mathfrak{g})$  generated by  $e_i, f_i$  and  $t_i, i \neq 0$  respectively.

In this theorem,  $z_i$  are formal variables and the functions on two sides have different expansion directions. Because the infinite formal power expansion is involved, we define two operators to be equal if the action of these two operators coincide on any vector in  $\Lambda$ .

**Proof.** The proof is basically the same as in the case of [Di]. First, we look at the matrix coefficients corresponding to the highest weight vectors of the two sides of the equalities above. With the precise expression of the correlation functions, we can see clearly that the left hand side is in the form of polynomials of  $z_1/z_2$  over  $(1 - z_1\xi/z_2)$ . With Theorem 1.1 in the last section, it is clear that the first term of the right hand side is the same as that of the other side, but in a different expansion direction. If we move the first term of the right side to the left hand side, we can explicitly calculate to show that this equality is valid. So we can prove that this statement is true for the coefficients corresponding to the highest weight vectors. However, we also know that  $F\delta(z_1\xi/z_2)$  and  $(z_1/z_2)^{1/2}F\delta(z_1\xi/z_2)$  are invariant under the action of the quantum affine algebra, due to the fact that  $F$  becomes an invariant subspace when  $z_1\xi = z_2$  and  $\delta(z_1/z_2)f(z_1, z_2) = \delta(z_1/z_2)f(z_1, z_1) = \delta(z_1/z_2)f(z_2, z_2)$  for any polynomial  $f(z_1, z_2)$ . Then, we know that  $F\delta(z_1\xi/z_2)$  and  $(z_1/z_2)^{1/2}F\delta(z_1\xi/z_2)$  are also intertwiners. So we can use the quantum affine algebra action to prove that this equality is valid for any matrix coefficient. Thus we prove the equality.

**Definition 2.1.** Quantum affine Clifford algebra of  $\mathbb{Z}$ -type is an associative algebra generated by  $\mathcal{U}(m), i \in I, m \in \mathbb{Z}$ , where  $I$  is defined in the Section 1. Let

$\Psi(z) = \sum_m \Psi_i(m) z^{-m} \otimes E_i$ , where  $E_i$  is a base for  $V^{(1)}$ . Then relations are

$$(2.2) \quad F(z_1/z_2)\Psi(z_2)\Psi(z_1) = G(z_1/z_2)\bar{R}^{(1,1)}(z_1/z_2)P\Psi(z_1)\Psi(z_2) + F\delta(z_1\xi/z_2).$$

Here  $F(z)$  and  $G(z)$  are the same as defined above.  $P$  is the permutation operator and  $R^{(1,1)}$  are on  $\mathbb{C}^N$  respectively.

**Definition 2.2.** Quantum affine Clifford algebra of  $\mathbb{Z} + 1/2$ -type is an associative algebra generated by  $\Psi_i(m)$ ,  $i \in J$ ,  $m \in \mathbb{Z} + 1/2$ , where  $J$  is defined in the Section 1 with  $N = 2n$ . Let  $\Psi(z) = \sum_m \Psi_i(m) z^{-m} \otimes v_i$ . Then relations are

$$(2.3) \quad F(z_1/z_2)\Psi(z_2)\Psi(z_1) = G(z_1/z_2)\bar{R}^{(1,1)}P(z_1/z_2)\Psi(z_1)\Psi(z_2) + (z_1/z_2)^{1/2}F\delta(z_1\xi/z_2).$$

Here  $F(z)$  and  $G(z)$  are the same as defined above.  $P$  is the permutation operator and  $R^{(1,1)}$  are on  $C^{2n}$  only.

In some sense, the algebras defined above are not rigorously defined, due to the fact that infinite expansions are involved. However, those definitions makes sense, if we look at a representation of this algebra on a space generated by one vector such that the operators of positive degrees are annihilators of this vector. We can also use the specific base as in [DvO] to specify those relations to define an ordered base, with which we can define an inverse limit type topology.

**Theorem 2.2.** Quantum affine Clifford algebra of  $\mathbb{Z}$ -type is isomorphic to the algebra generated by  $\Phi(z)$  for corresponding cases of type (A). Quantum affine Clifford algebra of  $\mathbb{Z} + 1/2$ -type is isomorphic to the algebra generated by  $\Phi(z)$  for corresponding cases of type (B).

Clearly what we really have to show here is that  $\Phi(z)$  gives faithful representations of those algebras, or we can say that all the relations between  $\Phi_j(n)$  are included in the commutation relations (A) or (B).

**Proof.** The proof is basically the same as in [Di]. We can use the base as in [DvO] to derive a specific basis for those quantum Clifford algebras, where the key observation is that  $z(1-1/z) = -1$ . This ensures a wedge type relations between

the generators that will show that this algebra has the wedge type character. By comparing characters, we can show that it is a faithful representation.

Let  $\Phi^*(z) = (C_+^{(1)} \otimes \text{id}) \cdot \Phi(z\xi^{-1})$ . The locations of the poles of the correlation functions of  $\Phi^2(z_1)\Phi^{*1}(z_2)$  clearly do not include the line  $z_1q^2 = z_2$ . From the commutation relations and the condition that  $|q| < 1$ , the multiplication of  $\Phi^*(z)$  and  $\Phi(zq^2)$  are well defined. Let  $E_i$ ,  $i \in J$ , be the base of  $V^{(1)}$  as defined in Section 1 and  $E_i^*$  be its right dual base. Thus we can say that  $(D_{q^2}\Phi_i)\bar{\Phi}_j^*E_j^* \otimes E_i = (1 \otimes 1 \otimes D_z^{-1})(1 \otimes \Phi(zq^2))\bar{\Phi}^*(z)$  is well defined. Here  $\Phi(z) = \sum E_i \otimes \Phi_i(z)$  and  $\Phi^*(z) = \sum E_i^* \otimes \Phi_i(z)$ .

Let  $\tilde{\mathfrak{L}}(z) = (1 \otimes \Phi(zq^2))\bar{\Phi}^*(z)$ . From [KKMMNN], we know that  $V^{(1)} \otimes \Lambda_i$  are irreducible. This shows that the dimension of space of operators  $X : V^{(1)} \otimes \Lambda \longrightarrow V_{q^2} \otimes \Lambda$ , which satisfy the relation :

$$(2.4) \quad X\Delta(a) = (D_q^2 \otimes 1)\Delta(a)X,$$

is 1 for the case of  $B_n^{(1)}$  and 2 for the case of  $D_n^{(1)}$ . Due to our normalization, it turns out that for the case of  $D_n^{(1)}$ , those two constants are the same, which can be determined by looking at their actions on the highest weight vectors.

**Theorem 2.3.**

$$(2.5) \quad \mathfrak{L}(z) = c(D_z \otimes 1)\tilde{\mathfrak{L}},$$

where  $c = \frac{\text{tr}(v_0, (D_{q^2}\Phi^*)\Phi v_0)}{\text{tr}(v_0, \mathfrak{L}(1)v_0)}$  and  $v_0$  is a highest weight vector of  $\Lambda$ .

We can remove the  $c$  in the formula above by normalizing  $\Phi(z)$  and  $\Phi^*(z)$  again.

Combining all the results above, it is clear that we can start from abstract algebra as in Definition 2.1 and 2.2, define the Fock space, which is generated by operators of negative degree and some zero degree operators, then the action of  $\mathfrak{L}$  can be derived. This gives the quantized spinor representations, which degenerate into classical constructions in [FF].

In [Di], we derive commutation for the quantized affine Clifford algebras coming from the spinor representations of  $U(\hat{\mathfrak{g}})$ . In the classical case [FF], they completely

coincide. It is interesting to compare those two algebras, which we believe are basically the same. In [B], one type of our spinor representations was obtained using vertex operator construction. It will be interesting to compare our constructions with that in [B], which should lead to another version of quantum Boson-Fermion correspondence, for which the result in [JKK] will be very useful. As we explain in the introduction, with the knowledge of corresponding q-KZ equations, we expect that all the constructions in [FF] can be derived in the same way. On the other hand, the further development of our theory should lead to the theory of quantization of vertex operator algebras[FLM], for which our new algebras should provide the proper example, which is closely related to the theory of formal factors in massive quantum field theory[Sm]. By now, our methods are used to construct representations, which are deformation of certain classical constructions. We do not know if it is possible to use this type of algebraic constructions to obtain representations, which are not known for the classical cases.

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